

Late time behaviors of the expanding universe in the IIB matrix model

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ABSTRACT: Recently we have studied the Lorentzian version of the IIB matrix model as a nonperturbative formulation of superstring theory. By Monte Carlo simulation, we have shown that the notion of time—as well as space—emerges dynamically from this model, and that we can *uniquely* extract the real-time dynamics, which turned out to be rather surprising: after some “critical time”, the $SO(9)$ rotational symmetry of the nine-dimensional space is spontaneously broken down to $SO(3)$ and the three-dimensional space starts to expand rapidly. In this paper, we study the same model based on the classical equations of motion, which are expected to be valid at later times. After providing a general prescription to solve the equations, we examine a class of solutions, which correspond to manifestly commutative space. In particular, we find a solution with an expanding behavior that naturally solves the cosmological constant problem.

KEYWORDS: Matrix Models, Superstring Vacua.

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1. Introduction

There are many fundamental questions in cosmology, which can, in principle, be answered by superstring theory. Describing the birth of the universe is one of the most fundamental ones. It is recognized, however, that the cosmic singularity is not resolved generally in perturbative string theory [1–5]. Therefore, in order to study the very early universe, we definitely need a nonperturbative formulation. Among various proposals [6–8] based on matrix models,¹ the IIB matrix model [7] looks most natural for describing the birth of the universe, since not only space but also time is expected to appear dynamically from the matrix degrees of freedom. Also the model is unique in that it is a manifestly covariant formulation, while the other proposals are based on the light-cone formulation, which breaks covariance.

¹For earlier attempts to apply matrix models to cosmology, see ref. [9–19].

One of the important issues in the IIB matrix model is to identify the configurations of matrices that dominate the path integral and to determine the corresponding space-time structure. This was studied by various approaches [20–34] in the Euclidean version of the model, which was shown to have finite partition function without any cutoffs [35, 36]. However, the Euclidean model does not seem suitable for cosmology since it does not provide the real-time dynamics. Furthermore, a recent study based on the gaussian expansion method suggests that the space-time obtained dynamically in the Euclidean model is three-dimensional rather than four-dimensional [34]. While this conclusion itself may have a profound implication, we do not know yet how it should be physically interpreted.

All these considerations led us to study the Lorentzian version of the IIB matrix model nonperturbatively [37]. By Monte Carlo simulation, we have shown that the Lorentzian model can be made well-defined nonperturbatively by first introducing infrared cutoffs and then removing them appropriately in the large- N limit. We have also found that the eigenvalue distribution of the matrix in the temporal direction extends in that limit, which implies that time emerges dynamically. (Supersymmetry of the model plays a crucial role.) Indeed we were able to extract a unique real-time dynamics, which turns out to have a surprising property. After some critical time, the $SO(9)$ rotational symmetry of the space is broken spontaneously down to $SO(3)$, and the three-dimensional space starts to expand rapidly. This result can be interpreted as the birth of the universe. Note that the concept of time-evolution also emerges dynamically in this model without ever having to specify the initial condition.

Although the length of time-evolution that has been extracted from Monte Carlo simulation is restricted due to finite N , we consider that the whole history of the universe can be obtained from the same model in the large- N limit. If this is true, we should be able to answer various important questions in both cosmology and particle physics. For instance, we may obtain the microscopic description of inflation. If we can reach the time at which stringy excitations and quantum gravitational effects become negligible, we may see how particles in the Standard Model (or possibly its extension) starts to appear. If we can study the behaviors of the model at much later times, we may be able to understand why the expansion of our universe is accelerating in the present epoch. Finally we may even predict how our universe will be like in the future.

While the late-time behaviors are difficult to study by direct Monte Carlo methods, the classical equations of motion are expected to become more and more valid at later times since the value of the action increases with the cosmic expansion. We will see that there are actually many classical solutions, which is reminiscent of the fact that string theory possesses infinitely many vacua that are perturbatively stable. However, unlike in perturbative string theory, we have the possibility to pick up the unique solution that describes our universe by requiring smooth connection to the behavior at earlier times accessible by Monte Carlo simulation. From this perspective, we consider it important to classify the classical solutions and to examine their cosmological implications. The aim of this paper is to make a first step in that direction. In particular, we find a classical solution with an expanding behavior that can naturally solve the cosmological constant problem.

Another issue we would like to address in this paper concerns how the commutative

space-time appears from this model. This is important since the $\text{SO}(9)$ symmetry breaking observed in the Monte Carlo simulation is understood intuitively by a mechanism, which relies crucially on the fact that the matrices that represent the space-time are noncommutative [37]. We show that classical solutions which correspond to manifestly commutative space can be easily constructed, and discuss the vanishing of noncommutativity between space and time in some simple examples. This implies that the emergence of commutative space-time is indeed possible in this model at later times.

The rest of this paper is organized as follows. In section 2 we briefly review the IIB matrix model and the results obtained by our previous Monte Carlo studies [37]. In section 3 we provide a general prescription to find classical solutions. In particular, we show that this can be done systematically by using Lie algebras. Then we restrict ourselves to manifestly space-space commutative solutions and present a complete classification of such solutions within certain simplifying ansatz. The particular solution discussed in our previous publication [38] also appears in this classification. In section 4 we obtain explicit time-evolution of the scale factor and the Hubble parameter for some simple solutions, and discuss their cosmological implications. Section 5 is devoted to a summary and discussions. In appendix A we review the irreducible unitary representations of the $\text{SU}(1,1)$ algebra, which will be needed in constructing the solutions discussed in section 4. In appendix B we list the other simple solutions which are not discussed in section 4. In appendix C we give some examples of solutions that are not manifestly space-space commutative.

2. The IIB matrix model and the birth of the universe

In this section we discuss why the IIB matrix model is considered as a nonperturbative formulation of type IIB superstring theory in ten dimensions, and how the results suggesting the birth of the universe were obtained by our previous Monte Carlo studies of this model.

The action of the IIB matrix model is given as [7]

$$S = S_b + S_f , \quad (2.1)$$

$$S_b = -\frac{1}{4g^2} \text{tr} \left([A_\mu, A_\nu][A^\mu, A^\nu] \right) , \quad (2.2)$$

$$S_f = -\frac{1}{2g^2} \text{tr} \left(\Psi_\alpha (\mathcal{C} \Gamma^\mu)_{\alpha\beta} [A_\mu, \Psi_\beta] \right) , \quad (2.3)$$

where A_μ ($\mu = 0, \dots, 9$) and Ψ_α ($\alpha = 1, \dots, 16$) are $N \times N$ Hermitian matrices. The Lorentz indices μ and ν are contracted using the metric $\eta = \text{diag}(-1, 1, \dots, 1)$. The 16×16 matrices Γ^μ are ten-dimensional gamma matrices after the Weyl projection, and the unitary matrix \mathcal{C} is the charge conjugation matrix. The action has manifest $\text{SO}(9,1)$ symmetry, where A_μ and Ψ_α transform as a vector and a Majorana-Weyl spinor, respectively.

There are various evidences that the model gives a nonperturbative formulation of superstring theory. First of all, the action (2.1) can be viewed as a matrix regularization of the worldsheet action of type IIB superstring theory in a particular gauge known as the Schild gauge [7]. It has also been argued that configurations of block-diagonal matrices correspond to a collection of disconnected worldsheets with arbitrary genus. Therefore,

instead of being equivalent just to the worldsheet theory, the large- N limit of the matrix model is expected to be a second-quantized theory of type IIB superstrings, which includes multi-string states. Secondly, D-branes are represented as classical solutions in the matrix model, and the interaction between them calculated at one loop reproduced correctly the known results from type IIB superstring theory [7]. Thirdly, one can derive the light-cone string field theory for the type IIB case from the matrix model [39] with a few assumptions. In the matrix model, one can define the Wilson loops, which can be naturally identified with the creation and annihilation operators of strings. Then, from the Schwinger-Dyson equations for the Wilson loops, one can actually obtain the string field Hamiltonian.

In all these connections to string theory, it is crucial that the model has two kinds of fermionic symmetries given as

$$\begin{cases} \delta^{(1)} A_\mu = i\epsilon_1 C\Gamma_\mu \Psi , \\ \delta^{(1)} \Psi = \frac{i}{2}\Gamma^{\mu\nu}[A_\mu, A_\nu]\epsilon_1 , \end{cases} \quad (2.4)$$

$$\begin{cases} \delta^{(2)} A_\mu = 0 , \\ \delta^{(2)} \Psi = \epsilon_2 \mathbb{1} , \end{cases} \quad (2.5)$$

where $\mathbb{1}$ is the unit matrix. It also has the bosonic symmetry given by

$$\begin{cases} \delta^{(3)} A_\mu = c_\mu \mathbb{1} , \\ \delta^{(3)} \Psi = 0 . \end{cases} \quad (2.6)$$

Let us denote the generators of (2.4), (2.5) and (2.6) by $Q^{(1)}$, $Q^{(2)}$ and P_μ , respectively, and define their linear combinations

$$\tilde{Q}^{(1)} = Q^{(1)} + Q^{(2)} , \quad \tilde{Q}^{(2)} = i(Q^{(1)} - Q^{(2)}) . \quad (2.7)$$

Then, we find that the generators satisfy the algebra

$$[\epsilon_1 C\tilde{Q}^{(i)}, \epsilon_2 C\tilde{Q}^{(j)}] = -2\delta^{ij}\epsilon_1 C\Gamma^\mu \epsilon_2 P_\mu , \quad (2.8)$$

where $i, j = 1, 2$. This is nothing but the ten-dimensional $\mathcal{N} = 2$ supersymmetry. It is known that field theories with this symmetry necessarily include gravity, which suggests that so does the IIB matrix model. When we identify (2.8) with the ten-dimensional $\mathcal{N} = 2$ supersymmetry, the symmetry (2.6) is identified with the translational symmetry in ten dimensions, which implies that the eigenvalues of A_μ should be identified with the coordinates of ten-dimensional space-time. This identification is consistent with the one adopted in stating the evidences listed in the previous paragraph, and shall be used throughout this paper as well.

An interesting feature of the IIB matrix model is that the space-time itself is treated as a part of dynamical degrees of freedom in the matrices. One can therefore try to identify the dominant configurations of matrices in the path integral and to determine the corresponding space-time structure. For that purpose, one needs to define the partition function as a finite matrix integral. This is nontrivial since the bosonic part of the action (2.2) is not bounded from below. By decomposing it into two terms

$$S_b = -\frac{1}{2g^2} \text{tr} (F_{0i})^2 + \frac{1}{4g^2} \text{tr} (F_{ij})^2 , \quad (2.9)$$

where we have defined Hermitian matrices $F_{\mu\nu} = i[A_\mu, A_\nu]$, we find that the first term is negative, whereas the second term is positive. A common way to overcome this problem is to make a Wick rotation, which amounts to making replacements $A_0 \mapsto iA_{10}$, $\Gamma^0 \mapsto -i\Gamma^{10}$. The Euclidean version of the model obtained in this way is well-defined nonperturbatively without any cutoffs [35,36]. The space-time structure has been studied by various approaches in this Euclidean model [20–34]. However, it is nontrivial whether the Wick rotation is valid in a theory including gravity [40,41]. Furthermore, the recent result based on the gaussian expansion method [34] suggests that the space-time appearing dynamically in the Euclidean model is three-dimensional rather than four-dimensional.

For these reasons, we studied the Lorentzian model nonperturbatively for the first time in ref. [37]. In order to make the partition function finite, we introduced the infrared cutoffs by imposing the following constraints

$$\begin{aligned} \frac{1}{N}\text{tr}(A_0)^2 &\leq \kappa \frac{1}{N}\text{tr}(A_i)^2, \\ \frac{1}{N}\text{tr}(A_i)^2 &\leq L^2, \end{aligned} \tag{2.10}$$

where κ and L are the cutoff parameters. We have shown by Monte Carlo simulation that these cutoffs can be removed in the large- N limit in such a way that the physical quantities scale. The resulting theory thus obtained has no parameters other than one scale parameter. This feature is precisely what one expects for nonperturbative string theory.

It turned out that not only space but also time emerges dynamically in this Lorentzian model. We found that the eigenvalue distribution of A_0 extends in the large- N limit. Here supersymmetry of the model plays a crucial role. If we omit fermions, the eigenvalue distribution has a finite extent. Furthermore, it turned out that one can extract the real-time dynamics by working in the $SU(N)$ basis which diagonalizes A_0 . We found that after a critical time, the $SO(9)$ symmetry is spontaneously broken down to the $SO(3)$, and three out of nine spatial directions start to expand. This can be interpreted as the birth of the universe. Note that the real-time dynamics is an emergent notion in this model, and we do not even have to specify the initial conditions. The above result is unique in that sense.

3. Classical solutions in the Lorentzian model

3.1 General prescription to find classical solutions

In this subsection we present a general prescription to find classical solutions in the Lorentzian model. It is important to take into account that the two cutoffs had to be introduced in order to make the model well-defined as we reviewed in the previous section. Since the inequalities (2.10) are actually saturated as is also seen by Monte Carlo simulation [37], we search for stationary points of the bosonic action S_b for fixed $\frac{1}{N}\text{tr}(A_0)^2$ and $\frac{1}{N}\text{tr}(A_i)^2$. Then we have to extremize the function

$$\tilde{S} = \text{tr} \left(-\frac{1}{4}[A_\mu, A_\nu][A^\mu, A^\nu] + \frac{\tilde{\lambda}}{2}(A_0^2 - \kappa L^2) - \frac{\lambda}{2}(A_i^2 - L^2) \right), \tag{3.1}$$

where λ and $\tilde{\lambda}$ are the Lagrange multipliers.

Differentiating (3.1) with respect to A_0 and A_i , we obtain²

$$-[A_0, [A_0, A_i]] + [A_j, [A_j, A_i]] - \lambda A_i = 0 , \quad (3.2)$$

$$[A_j, [A_j, A_0]] - \tilde{\lambda} A_0 = 0 , \quad (3.3)$$

respectively. Differentiating (3.1) with respect to $\tilde{\lambda}$ and λ , we obtain

$$\frac{1}{N} \text{tr}(A_0^2) = \kappa L^2 , \quad (3.4)$$

$$\frac{1}{N} \text{tr}(A_i^2) = L^2 , \quad (3.5)$$

respectively. Once we obtain a solution to eqs. (3.2) and (3.3), we can substitute them into (3.4) and (3.5) to determine λ and $\tilde{\lambda}$ as a function of κ and L .

Here we point out that the terms in eqs. (3.2) and (3.3) proportional to the Lagrange multipliers break the $\text{SO}(9,1)$ symmetry in general. In fact, we will see that there is a set of solutions with $\lambda = \tilde{\lambda}$, which do not suffer from this explicit breaking. However, we do not impose this condition from the outset in order to keep our analysis as general as possible.

In what follows, we consider solutions with

$$A_i = 0 \quad \text{for } i > d , \quad (3.6)$$

where $d \leq 9$. This is motivated from our observation in Monte Carlo simulation that A_i in the extra dimensions remain small when the three-dimensional space expands. From this point of view, one may think that we should choose $d = 3$. However, one can actually construct a solution with larger d by taking a direct sum of solutions with smaller d , say with $d = 1$, as we will see in 3.3.2. We therefore keep d arbitrary for the moment.

A general prescription to solve the equations of motion (3.2) and (3.3) within the ansatz (3.6) is given as follows. Let us first define a sequence of commutation relations

$$[A_i, A_j] = iC_{ij} , \quad (3.7)$$

$$[A_i, C_{jk}] = iD_{ijk} , \quad (3.8)$$

$$[A_0, A_i] = iE_i , \quad (3.9)$$

$$[A_0, E_i] = iF_i , \quad (3.10)$$

$$[A_i, E_j] = iG_{ij} , \quad \dots , \quad (3.11)$$

where $1 \leq i, j, k \leq d$ and the symbols on the right-hand side represent Hermitian operators newly defined. Then we determine the relationship among $A_0, A_i, C_{ij}, D_{ijk}, E_i, F_i, G_{ij}, \dots$ so that the equations of motion (3.2) and (3.3) and the Jacobi identities are satisfied. We obtain a Lie algebra in this way. Considering that all the operators are Hermitian, each unitary representation of the Lie algebra gives a classical solution.

²Classical solutions in the IIB matrix model have been studied in ref. [42] for $\lambda = \tilde{\lambda} = 0$. They are studied in the Euclidean version in ref. [43].

3.2 Lie algebra for manifestly commutative space

Since we expect that commutative space appears at later times, we restrict ourselves to solutions corresponding to manifestly commutative space in what follows. (See appendix C for examples of solutions without this restriction.) This implies that we impose

$$[A_i, A_j] = 0 . \quad (3.12)$$

Then we follow the general prescription described in section 3.1. In particular, we show that one can actually close the algebra with a finite number of generators by imposing a simple condition.

Let us consider the relationship among $A_0, A_i, C_{ij}, D_{ijk}, E_i, F_i, G_{ij}$ in (3.7) \sim (3.11). First of all, (3.12) implies that

$$C_{ij} = 0 , \quad D_{ijk} = 0 . \quad (3.13)$$

It is convenient to make the irreducible decomposition of a d -dimensional second-rank tensor G_{ij} in (3.11) as

$$G_{ij} = M_{ij} + N_{ij} + \frac{1}{d} \delta_{ij} H , \quad (3.14)$$

$$\text{where } M_{ij} = M_{ji} , \quad \sum_{i=1}^d M_{ii} = 0 , \quad (3.15)$$

$$N_{ij} = -N_{ji} . \quad (3.16)$$

From the equations of motion (3.2) and (3.3), we obtain

$$F_i = \lambda A_i , \quad (3.17)$$

$$H = \tilde{\lambda} A_0 , \quad (3.18)$$

where we have used (3.9), (3.10), (3.11), (3.12) and (3.14).

Next we consider the Jacobi identities

$$[A_0, [A_i, A_j]] + [A_i, [A_j, A_0]] + [A_j, [A_0, A_i]] = 0 , \quad (3.19)$$

$$[A_0, [A_i, E_j]] + [A_i, [E_j, A_0]] + [E_j, [A_0, A_i]] = 0 , \quad (3.20)$$

$$[A_i, [E_j, E_k]] + [E_j, [E_k, A_i]] + [E_k, [A_i, E_j]] = 0 , \quad (3.21)$$

$$[E_i, [A_j, A_k]] + [A_j, [A_k, E_i]] + [A_k, [E_i, A_j]] = 0 . \quad (3.22)$$

Using (3.9), (3.11), (3.12) and (3.14) in (3.19), we find

$$N_{ij} = 0 . \quad (3.23)$$

Using (3.17) and (3.18) in (3.20), we find

$$[A_0, M_{ij}] = 0 , \quad [E_i, E_j] = 0 . \quad (3.24)$$

Similarly, from (3.21) and (3.22), we obtain

$$[E_j, M_{ki}] - i\frac{\lambda\tilde{\lambda}}{d}\delta_{ki}A_j = [E_k, M_{ij}] - i\frac{\lambda\tilde{\lambda}}{d}\delta_{ij}A_k , \quad (3.25)$$

$$[A_j, M_{ik}] - i\frac{\tilde{\lambda}}{d}\delta_{ki}E_j = [A_k, M_{ij}] - i\frac{\tilde{\lambda}}{d}\delta_{ij}E_k , \quad (3.26)$$

respectively. We can easily verify that the Jacobi identities

$$[A_0, [E_i, E_j]] + [E_i, [E_j, A_0]] + [E_j, [A_0, E_i]] = 0 , \quad (3.27)$$

$$[A_i, [A_j, A_k]] + [A_j, [A_k, A_i]] + [A_k, [A_i, A_j]] = 0 , \quad (3.28)$$

$$[E_i, [E_j, E_k]] + [E_j, [E_k, E_i]] + [E_k, [E_i, E_j]] = 0 \quad (3.29)$$

are trivially satisfied, hence giving no new relations among the operators.

Now that all the Jacobi identities among A_0 , A_i and E_i are satisfied, let us move on to the Jacobi identities that include M_{ij} . In general we need to introduce some new operators, which appear from the commutator of M_{ij} and one of A_0 , A_i , E_i . One way to close the algebra without introducing new operators is to impose that M_{ij} is diagonal. Let us denote the diagonal elements as $M_i \equiv M_{ii}$, which satisfy

$$\sum_{i=1}^d M_i = 0 \quad (3.30)$$

due to the traceless condition (3.15). In what follows, we will see that A_0 , A_i , E_i and M_i form a Lie algebra.

First, we find that (3.25) and (3.26) lead to

$$[E_i, M_j] = i\frac{\lambda\tilde{\lambda}}{d}(1 - d\delta_{ij})A_i , \quad (3.31)$$

$$[A_i, M_j] = i\frac{\tilde{\lambda}}{d}(1 - d\delta_{ij})E_i . \quad (3.32)$$

Applying these commutation relations to the Jacobi identity

$$[M_i, [A_j, E_k]] + [A_j, [E_k, M_i]] + [E_k, [M_i, A_j]] = 0 , \quad (3.33)$$

we obtain

$$[M_i, M_j] = 0 . \quad (3.34)$$

To summarize, the commutation relations among A_0 , A_i , E_i and M_i are obtained as

$$\begin{aligned} [A_i, A_j] &= 0 , \quad [A_0, A_i] = iE_i , \quad [A_0, E_i] = i\lambda A_i , \\ [E_i, E_j] &= 0 , \quad [A_i, E_j] = i\delta_{ij} \left(\frac{\tilde{\lambda}}{d}A_0 + M_i \right) , \quad [A_0, M_i] = 0 , \\ [A_i, M_j] &= i\frac{\tilde{\lambda}}{d}(1 - d\delta_{ij})E_i , \quad [E_i, M_j] = i\frac{\lambda\tilde{\lambda}}{d}(1 - d\delta_{ij})A_i , \quad [M_i, M_j] = 0 , \end{aligned} \quad (3.35)$$

where $1 \leq i, j \leq d$ with $1 \leq d \leq 9$. It is straightforward to verify that all the remaining Jacobi identities including M_i are satisfied due to the commutation relations (3.35). Thus we obtain the Lie algebra (3.35), which gives a class of manifestly space-space commutative solutions.

3.3 Some simplifications of the Lie algebra

In this subsection we consider some special cases of (3.35), which corresponds to simple Lie algebras.

3.3.1 the case with $M_i = 0$ and $\tilde{\lambda} = 0$

First we point out that one can set $M_i = 0$ and $\tilde{\lambda} = 0$ consistently in eq. (3.35) for arbitrary d , which results in the Lie algebra

$$\begin{aligned} [A_i, A_j] &= 0, & [A_0, A_i] &= iE_i, & [A_0, E_i] &= i\lambda A_i, \\ [E_i, E_j] &= 0, & [A_i, E_j] &= 0. \end{aligned} \quad (3.36)$$

We can further simplify (3.36) by setting $E_i = \pm\sqrt{\lambda}A_i$, which yields³

$$[A_i, A_j] = 0, \quad [A_0, A_i] = \pm i\sqrt{\lambda}A_i. \quad (3.37)$$

The classical solutions obtained from this Lie algebra were studied in ref. [38].

3.3.2 the $d = 1$ case

The Lie algebra simplifies considerably in the $d = 1$ case. In this case, eq. (3.30) implies that $M_1 = 0$. Thus eq. (3.35) reduces to

$$[A_0, A_1] = iE, \quad [A_0, E] = i\lambda A_1, \quad [A_1, E] = i\tilde{\lambda}A_0, \quad (3.38)$$

where we define $E \equiv E_1$. Note that the $\tilde{\lambda} = 0$ case of (3.38) is identical to the $d = 1$ case of (3.36).

As we mentioned in section 3.1, we can use the $d = 1$ solutions (3.38) to construct new solutions representing a higher-dimensional space-time in the following way. For that, we note that the equations of motion (3.2) and (3.3) have $\text{SO}(9)$ symmetry. Rotating the solution (3.38) by an $\text{SO}(9)$ transformation, we obtain an equivalent solution, which has the i -th spatial matrix given by $r_i A_1$ ($i = 1, \dots, 9$) with $r_i^2 = 1$. Taking a direct sum of these solutions with various values of r_i , we obtain a new solution:

$$\begin{aligned} A'_0 &= A_0 \otimes \mathbb{I}_k, \\ A'_i &= A_1 \otimes \text{diag}(r_i^{(1)}, r_i^{(2)}, \dots, r_i^{(k)}), \end{aligned} \quad (3.39)$$

$$\text{where } r_i^{(m)2} = 1 \quad (m = 1, \dots, k), \quad (3.40)$$

and \mathbb{I}_k is the $k \times k$ unit matrix. In particular, we can construct an $\text{SO}(D)$ symmetric solution by requiring that $r^{(m)}$'s be distributed uniformly on a unit S^{D-1} , where $1 \leq D \leq 9$. The $D = 4$ case would then be a physically interesting solution which represent $(3 + 1)$ -dimensional space-time with $R \times S^3$ geometry.

³The Lie algebra (3.37) in the $d = 3$ and $d = 4$ cases correspond to $A_{4,5}^{ab}$ and $A_{5,7}^{abc}$ in Table I of ref. [44], respectively.

Let us consider the case in which $\lambda \neq 0$ and $\tilde{\lambda} \neq 0$. In this case, the Lie algebra (3.38) can be identified either with the $SU(1, 1)$ algebra (the $SL(2, R)$ algebra)

$$[T_0, T_1] = iT_2, \quad [T_2, T_0] = iT_1, \quad [T_1, T_2] = -iT_0, \quad (3.41)$$

or with the $SU(2)$ algebra

$$[L_1, L_2] = iL_3, \quad [L_2, L_3] = iL_1, \quad [L_3, L_1] = iL_2, \quad (3.42)$$

depending on the signs of λ and $\tilde{\lambda}$ as follows.⁴

(a) $\lambda > 0$ and $\tilde{\lambda} > 0$: $SU(1, 1)$ algebra

$$\begin{aligned} A_0 &= aT_2, \quad A_1 = bT_0, \quad E = cT_1, \\ \lambda &= a^2, \quad \tilde{\lambda} = b^2, \quad ab = c. \end{aligned} \quad (3.43)$$

(b) $\lambda < 0$ and $\tilde{\lambda} < 0$: $SU(1, 1)$ algebra

$$\begin{aligned} A_0 &= aT_0, \quad A_1 = bT_1, \quad E = cT_2, \\ \lambda &= -a^2, \quad \tilde{\lambda} = -b^2, \quad ab = c. \end{aligned} \quad (3.44)$$

(c) $\lambda > 0$ and $\tilde{\lambda} < 0$: $SU(1, 1)$ algebra

$$\begin{aligned} A_0 &= aT_2, \quad A_1 = bT_1, \quad E = cT_0, \\ \lambda &= a^2, \quad \tilde{\lambda} = -b^2, \quad ab = c. \end{aligned} \quad (3.45)$$

(d) $\lambda < 0$ and $\tilde{\lambda} > 0$: $SU(2)$ algebra

$$\begin{aligned} A_0 &= aL_3, \quad A_1 = bL_1, \quad E = cL_2, \\ \lambda &= -a^2, \quad \tilde{\lambda} = b^2, \quad ab = c. \end{aligned} \quad (3.46)$$

The cases in which $\lambda = 0$ or $\tilde{\lambda} = 0$ are discussed in appendix B. Thus we find that the solutions with $d = 1$ are classified into the nine cases; namely, (a)~(d) in this subsection and (i)~(v) in appendix B. In section 4 we discuss cosmological implications of the above four cases (a)~(d).

3.3.3 the $d = 2$ case

As a more complicated example, we discuss the $d = 2$ case of (3.35). Since this case will not be discussed further in this paper, impatient readers may jump into section 4.

Let us note that eq. (3.30) gives $M_1 = -M_2 \equiv M$. For $\lambda \neq 0$ and $\tilde{\lambda} \neq 0$, we can rescale A_0, A_i, E_i and M appropriately so that eq. (3.35) can be rewritten as

$$[A_1, A_2] = [E_1, E_2] = [A_1, E_2] = [A_2, E_1] = [A_0, M] = 0,$$

⁴These two algebras appear in Table I of ref. [44] as $A_{3,8}$ and $A_{3,9}$, respectively.

$$\begin{aligned}
[A_0, A_1] &= iE_1, \quad [A_0, A_2] = iE_2, \quad [A_0, E_1] = i\text{sign}(\lambda)A_1, \quad [A_0, E_2] = i\text{sign}(\lambda)A_2, \\
[A_1, E_1] &= 2i(\text{sign}(\tilde{\lambda})A_0 + M), \quad [A_2, E_2] = 2i(\text{sign}(\tilde{\lambda})A_0 - M), \\
[A_1, M] &= -i\text{sign}(\tilde{\lambda})E_1, \quad [A_2, M] = i\text{sign}(\tilde{\lambda})E_2, \\
[E_1, M] &= -i\text{sign}(\lambda\tilde{\lambda})A_1, \quad [E_2, M] = i\text{sign}(\lambda\tilde{\lambda})A_2,
\end{aligned} \tag{3.47}$$

where $\text{sign}(\cdot)$ represents the sign function. We compare the above algebra with the $\text{SO}(2, 2)$ algebra and $\text{SO}(4)$ algebra, which can be expressed in a unified way as

$$[L_{\alpha\beta}, L_{\gamma\delta}] = ig_{\alpha\gamma}L_{\beta\delta} + ig_{\beta\delta}L_{\alpha\gamma} - ig_{\alpha\delta}L_{\beta\gamma} - ig_{\beta\gamma}L_{\alpha\delta}, \tag{3.48}$$

where $\alpha, \beta, \gamma, \delta = 1, 2, 3, 4$ and $g_{\alpha\beta}$ represents the metric. By making an identification

$$\begin{aligned}
A_0 &= L_{23}, \\
A_1 &= L_{12} + L_{34}, \quad A_2 = L_{12} - L_{34}, \\
E_1 &= g_{22}L_{13} - g_{33}L_{24}, \quad E_2 = g_{22}L_{13} + g_{33}L_{24}, \\
M &= -\text{sign}(\lambda)L_{14},
\end{aligned} \tag{3.49}$$

it is easy to check that eq. (3.47) is satisfied if

$$(g_{\alpha\beta}) = \begin{cases} \text{diag}(1, 1, -1, -1) & \text{for } \lambda > 0, \tilde{\lambda} > 0, \\ \text{diag}(1, -1, -1, 1) & \text{for } \lambda < 0, \tilde{\lambda} < 0, \\ \text{diag}(1, -1, 1, -1) & \text{for } \lambda > 0, \tilde{\lambda} < 0, \\ \text{diag}(1, 1, 1, 1) & \text{for } \lambda < 0, \tilde{\lambda} > 0. \end{cases} \tag{3.50}$$

The first three cases correspond to the $\text{SO}(2, 2)$ algebra, whereas the last one corresponds to the $\text{SO}(4)$ algebra. A unitary representation of these algebras corresponds to a classical solution. More in-depth studies of these solutions are left for future investigations.

4. Cosmological implications of some simple classical solutions

In this section we discuss the cosmological implications of the $\text{SU}(1, 1)$ solutions (a)~(c) and the $\text{SU}(2)$ solution (d) discussed in section 3.3.2. As a warming up, we start with the $\text{SU}(2)$ solution, which is simpler, and then move on to the $\text{SU}(1, 1)$ solutions, which exhibit physically more interesting behaviors.

4.1 Solutions based on the $\text{SU}(2)$ algebra — a warm-up

Let us consider the solution (3.46) based on the $\text{SU}(2)$ algebra (3.42). As we mentioned in section 3.1, we use the unitary representations. The irreducible unitary representations of the $\text{SU}(2)$ algebra is specified by their spins J , which are non-negative integers or half-integers. In the spin J representation, the matrix elements of the generators in eq. (3.42) are given by

$$(L_1)_{mn} = \frac{1}{2}\sqrt{(J-n)(J+n+1)}\delta_{m,n+1} + \frac{1}{2}\sqrt{(J+n)(J-n+1)}\delta_{m,n-1},$$

$$\begin{aligned}
(L_2)_{mn} &= \frac{1}{2i} \sqrt{(J-n)(J+n+1)} \delta_{m,n+1} - \frac{1}{2i} \sqrt{(J+n)(J-n+1)} \delta_{m,n-1} , \\
(L_3)_{mn} &= n \delta_{mn} ,
\end{aligned} \tag{4.1}$$

where $-J \leq m, n \leq J$. From eq. (3.46), we find that A_0 is diagonal, whereas A_1 has a tri-diagonal structure. This motivates us to extract the time evolution of space from the 3×3 submatrices of A_0 and A_1 defined as [37]

$$\bar{A}_0(n) = a \begin{pmatrix} n-1 & 0 & 0 \\ 0 & n & 0 \\ 0 & 0 & n+1 \end{pmatrix} , \tag{4.2}$$

$$\bar{A}_1(n) = \frac{b}{2} \begin{pmatrix} 0 & \sqrt{(J+n)(J-n+1)} & 0 \\ \sqrt{(J+n)(J-n+1)} & 0 & \sqrt{(J-n)(J+n+1)} \\ 0 & \sqrt{(J-n)(J+n+1)} & 0 \end{pmatrix} , \tag{4.3}$$

where $-J+1 \leq n \leq J-1$.

Similarly, we consider an $\text{SO}(4)$ symmetric solution (3.39), where $r_i^{(m)2} = 1$ ($m = 1, \dots, k$) and $r^{(m)}$ are uniformly distributed on a unit S^3 . In this case, we define

$$\begin{aligned}
\bar{A}'_0(n) &= \bar{A}_0(n) \otimes 1_k , \\
\bar{A}'_i(n) &= \bar{A}_1(n) \otimes \text{diag}(r_i^{(1)}, \dots, r_i^{(k)}) ,
\end{aligned} \tag{4.4}$$

where $\bar{A}'_i(n)$ represents the structure of space at a discrete time n . We also define the extent of space at a discrete time n by

$$R(n) \equiv \sqrt{\frac{1}{3k} \text{tr}(\bar{A}'_1(n))^2} = \sqrt{\frac{b^2}{3} (J(J+1) - n^2)} . \tag{4.5}$$

Let us then discuss the continuum limit, in which we send $a \rightarrow 0$ and $J \rightarrow \infty$, and define the continuum time by $t = na$. We see from eq. (4.5) that a nontrivial dependence of R on t is obtained by keeping $t_{\max} = Ja$ and $\frac{b}{a} = \alpha$ fixed. This leads to

$$R(t) = R_{\max} \sqrt{1 - \left(\frac{t}{t_{\max}} \right)^2} , \quad \text{where } R_{\max} = \frac{\alpha t_{\max}}{\sqrt{3}} . \tag{4.6}$$

From the fact that A_0 is diagonal and A_1 has a tri-diagonal structure, we consider that the space-time noncommutativity vanishes in the continuum limit. Let us also note that the extents of space and time defined by (3.4) and (3.5) are given by $L \sim \alpha t_{\max}$ and $\kappa \sim 1/\alpha$.

Let us next discuss the cosmological implication of this solution. For that we naively identify $R(t)$ with the scale factor in the Friedman-Robertson-Walker metric. Then the space-time is $R \times S^3$, where the radius of S^3 is given by $R(t)$. Figure 1 shows the time dependence of $R(t)$. The universe expands towards $t = 0$ and shrinks after $t = 0$. The Hubble parameter can be defined in terms of the scale factor $R(t)$ as

$$H(t) = \frac{\dot{R}(t)}{R(t)} = c R(t)^{-\frac{3}{2}(1+w)} , \tag{4.7}$$

where c is a constant. From this, we can evaluate the parameter w using

$$w = -\frac{2R(t)}{3} \frac{d \log H(t)}{dR(t)} - 1 . \quad (4.8)$$

Let us recall that $w = 1/3$, $w = 0$ and $w = -1$ correspond to the radiation dominated universe, the matter dominated universe and the cosmological constant term, respectively.

In the present case of SU(2) solution, we find from (4.6) that

$$H = \frac{R_{\max}}{t_{\max} R^2} \sqrt{R_{\max}^2 - R^2} , \quad (4.9)$$

$$w = \frac{2t_{\max}^2}{3t^2} - \frac{1}{3} \quad (4.10)$$

for $t < 0$. The parameter w is $w = 1/3$ at $t = -t_{\max}$, and it diverges as one approaches $t = 0$.

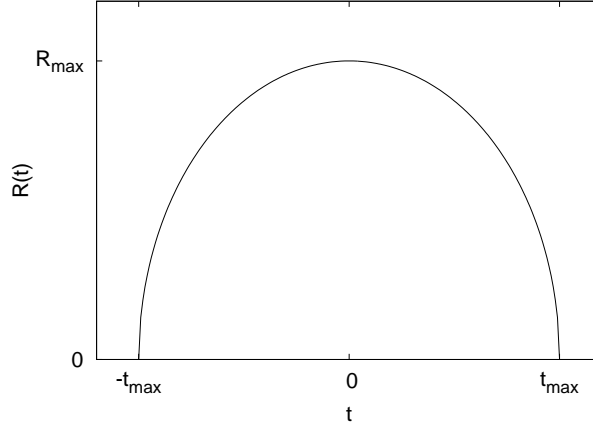


Figure 1: The time dependence of $R(t)$ in the SU(2) solution

4.2 Solutions based on the SU(1,1) algebra

Next we discuss the solutions (a), (b) and (c) in section 3.3.2, which are based on the SU(1,1) algebra (3.41). We construct SO(4) symmetric solutions for these cases as in eq. (3.39). The irreducible unitary representations of the SU(1,1) algebra are summarized in appendix A. Apart from the trivial representation, there are three types: the primary unitary series representation (PUSR) (A.4), the discrete series representation (DSR) (A.6) and (A.7), the complementary unitary series representation (CUSR) (A.5). In all these representations, the generator T_0 is diagonal. Hence, we can analytically treat the case (b), in which A_0 is proportional to T_0 , while we need to diagonalize A_0 numerically in the cases (a) and (c). For this reason, we start with the case (b). In this case, we find from (A.3) that A_1 has a tri-diagonal structure. Therefore, we extract 3×3 submatrices $\bar{A}_0(n)$ and $\bar{A}_1(n)$ similarly to the SU(2) case discussed in section 4.1. In what follows, we discuss the solutions for each representation separately.

Let us first discuss the solutions corresponding to PUSR. $\bar{A}_0(n)$ and \bar{A}_1 takes the form

$$\bar{A}_0(n) = a \begin{pmatrix} n-1+\epsilon & 0 & 0 \\ 0 & n+\epsilon & 0 \\ 0 & 0 & n+1+\epsilon \end{pmatrix}, \quad (4.11)$$

$$\bar{A}_1(n) = \frac{ib}{2} \begin{pmatrix} 0 & n+i\rho-\frac{1}{2}+\epsilon & 0 \\ -n+i\rho+\frac{1}{2}-\epsilon & 0 & n+i\rho+\frac{1}{2}+\epsilon \\ 0 & -n+i\rho-\frac{1}{2}-\epsilon & 0 \end{pmatrix}, \quad (4.12)$$

where $\epsilon = 0$ or $\frac{1}{2}$ and ρ is a non-negative number, which specifies a representation.

When we consider an SO(4) symmetric solution, we define $\bar{A}'_0(n)$ and $\bar{A}'_i(n)$ as in eq. (4.4). Then we find that the extent of space $R(n)$ at a discrete time n becomes

$$R(n) = \sqrt{\frac{b^2}{3} \left(n^2 + \rho^2 + \frac{1}{4} \right)}. \quad (4.13)$$

Let us take the continuum limit. We define the continuum time by $t = na$ and take the limit in which $a \rightarrow 0$ with $\frac{b}{a} = \alpha$ fixed. We can tune ρ so that $t_0 \equiv \rho a$ is fixed. Then $R(t)$ is given by

$$R(t) = \sqrt{\frac{\alpha^2}{3} (t^2 + t_0^2)}. \quad (4.14)$$

As in the SU(2) case, the continuum limit of this solution represents a commutative (3+1)-dimensional space-time. In order to evaluate L and κ in (3.4) and (3.5), we introduce a cutoff N for the dimension of the representation. Then, it is easy to see that (3.4) and (3.5) give $L \sim Na$ and $\kappa \sim 1/\alpha$. The infrared cutoff L is removed by sending N to infinity faster than $1/a$. The other parameter κ , which corresponds to the ratio of $\text{tr}(A_0)^2/N$ to $\text{tr}(A_i)^2/N$, is finite in the continuum limit, which looks different from the situation encountered in Monte Carlo studies, where we had to send κ to infinity [37]. This is not so surprising, however, given that we are looking at different time regions, and the speed of expansion changes qualitatively depending on which region we are looking at.

Here we naively identify $R(t)$ with the scale factor again. Then the space-time is $R \times S^3$, where the radius $R(t)$ of S^3 is time dependent. From eq. (4.14), we obtain the Hubble parameter H and the parameter w as

$$H = \frac{\alpha}{\sqrt{3}R^2} \sqrt{R^2 - \frac{\alpha^2 t_0^2}{3}}, \quad (4.15)$$

$$w = -\frac{2t_0^2}{3t^2} - \frac{1}{3}, \quad (4.16)$$

We find that w converges to $-\frac{1}{3}$ as $t \rightarrow \infty$, which corresponds to the expansion of universe with a constant velocity.

If we identify t_0 with the present time, the present value of w is -1 . This value of w corresponds to the cosmological constant, which explains the present accelerating expansion

of the universe. Moreover, the corresponding cosmological constant becomes of the order of $(1/t_0)^4$, which naturally solves the cosmological constant problem. As we mentioned above, w increases with time and approaches $-\frac{1}{3}$. This means that the cosmological constant actually vanishes in the future. In fig. 2 we show the time dependence of $R(t)$ and the parameter w .

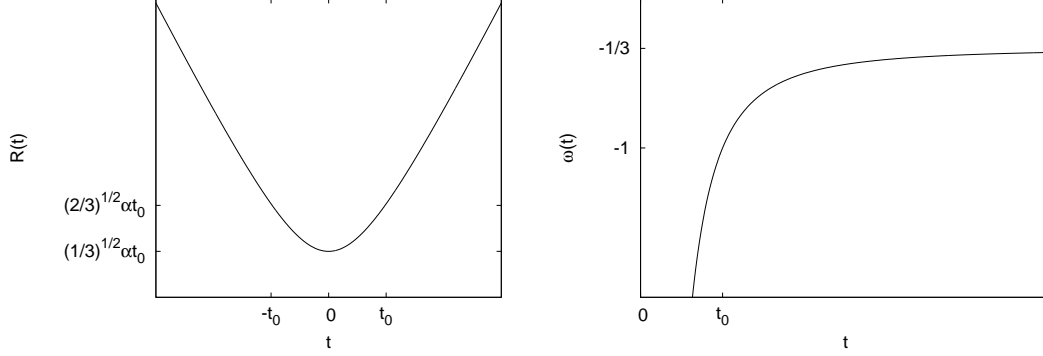


Figure 2: The time dependence of the scale factor $R(t)$ (Left) and the parameter w (Right) in the SU(1,1) solution with the PUSR.

Let us next discuss the solutions corresponding to DSR. $\bar{A}_0(n)$ takes the form (4.11), where $\epsilon = 0$ or $\frac{1}{2}$, and \bar{A}_1 takes the form

$$\bar{A}_1(n) = \frac{ib}{2} \begin{pmatrix} 0 & \sqrt{(n+\tau)(n-\tau-1)} & 0 \\ -\sqrt{(n+\tau)(n-\tau-1)} & 0 & \sqrt{(n-\tau)(n+\tau+1)} \\ 0 & -\sqrt{(n-\tau)(n+\tau+1)} & 0 \end{pmatrix}, \quad (4.17)$$

with $\tau = -1, -2, -3, \dots$ for $\epsilon = 0$ and $\tau = -\frac{1}{2}, -\frac{3}{2}, -\frac{5}{2}, \dots$ for $\epsilon = \frac{1}{2}$. There is a constraint $n \leq -\tau - \epsilon + 1$ or $n \leq \tau - \epsilon - 1$. Then $R(n)$ is given by

$$R(n) = \sqrt{\frac{b^2}{3}(n^2 - \tau(\tau+1))}. \quad (4.18)$$

In the continuum limit, we can tune τ so that $t_0 \equiv a\tau(\tau+1)$ is fixed. Then $R(t)$ is given by

$$R(t) = \sqrt{\frac{\alpha^2}{3}(t^2 - t_0^2)}, \quad (4.19)$$

where the range of t is restricted to either $t \geq t_0$ or $t \leq -t_0$.

From eq. (4.19), we obtain the Hubble parameter H and the parameter w as

$$H = \frac{\alpha}{\sqrt{3}R^2} \sqrt{R^2 + \frac{\alpha^2 t_0^2}{3}}, \quad (4.20)$$

$$w = \frac{2t_0^2}{3t^2} - \frac{1}{3} \quad (4.21)$$

for $t \geq t_0$. We find that the parameter w becomes $w = 1/3$ at $t = t_0$ and $w = 0$ at $t = \sqrt{2}t_0$. The former corresponds to the radiation dominant universe, while the latter to the matter dominant universe. Thus this solution may represent some part of the history of the universe. Figure 3 shows the time dependence of $R(t)$ and the parameter w , respectively.

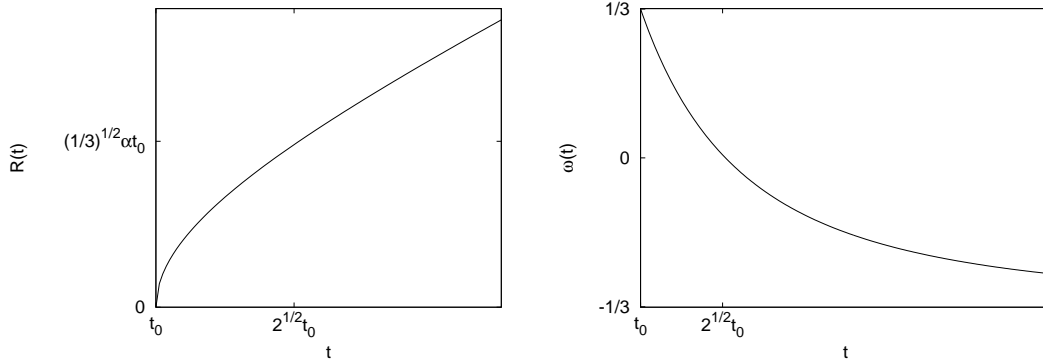


Figure 3: The time dependence of the scale factor $R(t)$ (Left) and the parameter w (Right) in the $SU(1, 1)$ solution with the DSR.

Next we discuss the solutions corresponding to CUSR. $\bar{A}_0(n)$ takes the form (4.11), where $\epsilon = 0$, and $\bar{A}_1(n)$ takes the form (4.17), where τ within $-1 < \tau < 0$ specifies a representation. The extent of space $R(n)$ is given by eq. (4.18). In the continuum limit, we obtain (4.19), where $t_0 = 0$.

Finally, we discuss the cases (a) and (c) in section 3.3.2. In these cases, we have diagonalized A_0 numerically. The physical consequence obtained from (a) is essentially the same as (b). On the other hand, no accelerating expansion is obtained for the case (c). $R(t)$ has two peaks, one in the $t < 0$ region and the other in the $t > 0$ region, and the minimum lies at $t = 0$.

5. Summary and discussions

In this paper we studied the late time behaviors of the universe in the Lorentzian version of the IIB matrix model. We investigated the classical equations of motion, which are expected to be valid at later times. This is a complementary approach to Monte Carlo simulation,⁵ which was used previously to study the birth of the universe in the same model. First we provided a general prescription to solve the equations of motion. The problem reduces to that of finding a unitary representation of a Lie algebra.

⁵See ref. [45] for a recent review on Monte Carlo studies of matrix models and supersymmetric gauge theories in the context of string theory.

In this way, we obtained a class of solutions that are manifestly space-space commutative. The simplest ones in this class are the $d = 1$ solutions with $A_2 = \dots = A_9 = 0$, from which we can easily construct the ones representing higher dimensional space-time as well. We made a complete classification of such solutions. Some solutions represent expanding $(3 + 1)$ -dimensional universe without space-time noncommutativity in the continuum limit. In particular, we find that there exists a solution, in which the parameter w changes smoothly from -1 to $-1/3$. This explains why we seem to have a tiny cosmological constant in the present epoch, and hence can naturally solve the cosmological constant problem. While we do not insist that this particular solution really describes our universe, we consider that the cosmological constant problem can be naturally solved in the Lorentzian matrix model in a similar manner.

Corresponding to what we have done in Monte Carlo simulation, we have introduced infrared cutoffs in both the temporal and spatial directions. These are represented by the Lagrange multipliers λ and $\tilde{\lambda}$ introduced in the action (3.1). In general, this breaks the $SO(9,1)$ symmetry of the model explicitly. Let us note, however, that it is possible to have $\lambda = \tilde{\lambda}$ in the cases (a) and (b) in section 3.3.2. Such solutions break the $SO(9,1)$ symmetry *spontaneously*. We expect that the explicit breaking of the Lorentz symmetry by the infrared cutoffs disappears in the large- N limit. If that is really the case, we should select a solution with $\lambda = \tilde{\lambda}$. It is intriguing to note that the cases (a) and (b) are indeed the ones that are physically interesting.

The $(3+1)$ -dimensional space-time represented by the solutions discussed mainly in this paper has the topology $R \times S^3$. This is a restriction which we have as long as we construct such solutions based on the $d = 1$ solution. In other constructions, we can also obtain solutions representing a space with the topology of a three-dimensional ball as we discussed in appendix B. While the Monte Carlo results seem to be more consistent with the latter topology of the space, it remains to be seen what kind of topology is actually realized at later times.

Below we list some directions for future investigations.

First we consider it important to examine the stability of the solutions we found in this paper. It would be also interesting to calculate the one-loop effective action around the solutions. That would tell us the validity of the solutions, and we should be able to know how late the time should be for the solutions to be valid.

Secondly it is important to understand better how one should extract the information of the space-time metric from a matrix configuration. Ref. [46] shows that this is indeed possible, in principle, if one interprets the matrix as a covariant derivative on the space-time manifold, where the general coordinate invariance is realized manifestly as a subgroup of the $SU(N)$ symmetry. However, this interpretation is different from the one adopted in this paper, which is compatible with the supersymmetry as we reviewed in section 2. The precise relationship between the two interpretations is yet to be clarified, although it is tempting to consider that they are related to each other by T-duality of type IIB superstring theory. In this work we have naively identified $R(t)$ with the scale factor in the Friedman-Robertson-Walker metric when we discuss cosmological implications in section 4. It remains to be seen whether this identification can somehow be justified.

Thirdly we consider it important to study a wider class of solutions using the general prescription provided in this paper. In particular, it would be interesting to examine the solutions, which are not manifestly space-space commutative, based on the Lie algebra (3.49) or the one given in appendix C. Also it would be interesting to investigate solutions with nontrivial structure in the extra dimensions. Such structure is expected to play a crucial role [47, 48] in determining the matter content at late times and in finding how the standard model appears from the matrix model. Eventually, we have to single out the solution, which is smoothly connected to the unique result at earlier times accessible by Monte Carlo simulation.

Developments in the above directions would enable us to solve various fundamental problems in particle physics and cosmology. For instance, we should be able to understand the mechanism of inflation and to clarify what the dark matter and the dark energy are. We hope that the present work will trigger such developments.

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A. Unitary representations of the $SU(1, 1)$ algebra

In this appendix we summarize the unitary representations of the $SU(1, 1)$ algebra (3.41) based on ref. [49]. The generators are realized in the space of square integrable functions in the region $[0, 2\pi]$, which we denote as $\mathcal{L}^2(0, 2\pi)$ in what follows. They are given as

$$\begin{aligned}\mathcal{T}_0 &= i\frac{d}{d\theta} + \epsilon, \\ \mathcal{T}_1 &= \frac{i}{2} \left[(\tau + \epsilon)e^{i\theta} + (\tau - \epsilon)e^{-i\theta} - 2\sin\theta \frac{d}{d\theta} \right], \\ \mathcal{T}_2 &= \frac{1}{2} \left[-(\tau + \epsilon)e^{i\theta} + (\tau - \epsilon)e^{-i\theta} - 2i\cos\theta \frac{d}{d\theta} \right],\end{aligned}\tag{A.1}$$

where $0 \leq \theta < 2\pi$, and $\tau \in \mathbb{C}$ and $\epsilon \in \mathbb{R}$ are parameters. It is easy to verify that these operators satisfy the $SU(1, 1)$ algebra

$$[\mathcal{T}_0, \mathcal{T}_1] = i\mathcal{T}_2, \quad [\mathcal{T}_2, \mathcal{T}_0] = i\mathcal{T}_1, \quad [\mathcal{T}_1, \mathcal{T}_2] = -i\mathcal{T}_0.\tag{A.2}$$

Taking the set of functions $\{e^{-im\theta}; m \in \mathbb{Z}\}$ as a basis of $\mathcal{L}^2(0, 2\pi)$, one obtains the matrix elements of the generators as

$$(\mathcal{T}_0)_{mn} = (\epsilon + n)\delta_{mn},$$

$$\begin{aligned}
(\mathcal{T}_1)_{mn} &= -\frac{i}{2}(n - \tau + \epsilon)\delta_{m,n+1} + \frac{i}{2}(n + \tau + \epsilon)\delta_{m,n-1} , \\
(\mathcal{T}_2)_{mn} &= -\frac{1}{2}(n - \tau + \epsilon)\delta_{m,n+1} - \frac{1}{2}(n + \tau + \epsilon)\delta_{m,n-1} ,
\end{aligned} \tag{A.3}$$

where $m, n \in \mathbb{Z}$.

The unitary irreducible representations are classified as follows. We denote the matrix elements of T_μ in (3.41) by $(T_\mu)_{mn}$, which differs from $(\mathcal{T}_\mu)_{mn}$, in general, due to some factors introduced to define the scalar product that realizes the unitarity. Since $SU(1,1)$ is a noncompact group, all the nontrivial unitary representations are infinite dimensional.

1) primary unitary series representations

$$\begin{aligned}
\tau &= i\rho - \frac{1}{2} \quad (\rho \in \mathbb{R}_{\geq 0}) , \quad \epsilon = 0 \quad \text{or} \quad \frac{1}{2} , \\
(T_\mu)_{mn} &= (\mathcal{T}_\mu)_{mn} .
\end{aligned} \tag{A.4}$$

2) complementary unitary series representations

$$\begin{aligned}
-1 < \tau < 1 , \quad \epsilon = 0 , \\
(T_\mu)_{mn} &= \left(\frac{\Gamma(\tau - m + 1)\Gamma(-\tau - n)}{\Gamma(\tau - n + 1)\Gamma(-\tau - m)} \right)^{\frac{1}{2}} (\mathcal{T}_\mu)_{mn} .
\end{aligned} \tag{A.5}$$

3) discrete series representations (I)

$$\begin{aligned}
\tau &= -1, -2, -3, \dots , \quad \epsilon = 0 , \\
\text{or} \\
\tau &= -\frac{1}{2}, -\frac{3}{2}, -\frac{5}{2}, \dots , \quad \epsilon = \frac{1}{2} , \\
(T_\mu)_{mn} &= \left(\frac{\Gamma(\tau + m + \epsilon + 1)\Gamma(-\tau + n + \epsilon)}{\Gamma(\tau + n + \epsilon + 1)\Gamma(-\tau + m + \epsilon)} \right)^{\frac{1}{2}} (\mathcal{T}_\mu)_{mn} , \\
m, n &\geq -\tau - \epsilon .
\end{aligned} \tag{A.6}$$

4) discrete series representations (II)

$$\begin{aligned}
\tau &= -1, -2, -3, \dots , \quad \epsilon = 0 , \\
\text{or} \\
\tau &= -\frac{1}{2}, -\frac{3}{2}, -\frac{5}{2}, \dots , \quad \epsilon = \frac{1}{2} , \\
(T_\mu)_{mn} &= \left(\frac{\Gamma(\tau - m - \epsilon + 1)\Gamma(-\tau - n - \epsilon)}{\Gamma(\tau - n - \epsilon + 1)\Gamma(-\tau - m - \epsilon)} \right)^{\frac{1}{2}} (\mathcal{T}_\mu)_{mn} , \\
m, n &\leq \tau - \epsilon .
\end{aligned} \tag{A.7}$$

5) trivial representation

$$T_\mu = 0 . \tag{A.8}$$

B. The other classical solutions in the $d = 1$ case

In this appendix we discuss the solutions based on the Lie algebra (3.38) obtained for $d = 1$ when $\lambda = 0$ or $\tilde{\lambda} = 0$. The solutions can be classified into the following five cases.⁶

An interesting feature of the $\tilde{\lambda} = 0$ case is that we can construct solutions representing higher dimensional space-time with topology other than S^{D-1} . The reason is that we can rescale A_1 and E in (3.38) without changing λ when $\tilde{\lambda} = 0$. Therefore, we do not need to impose the condition (3.40) in constructing the new solution (3.39). For instance, we can distribute $r_i^{(m)}$ uniformly in a three-dimensional ball B^3 so that the solution represents a $(3 + 1)$ -dimensional space-time with $SO(3)$ symmetry.

(i) $\lambda = 0$ and $\tilde{\lambda} = 0$

The nontrivial irreducible representations are parametrized by $a \in \mathbb{R}$ ($a \neq 0$). A_0 and A_1 are given by the operators acting on the space of functions of x with L^2 integrability, which is denoted by $\mathcal{L}^2(\mathbb{R})$ in what follows. The operators are given explicitly as

$$A_0 = -ia\sqrt{\lambda}\frac{d}{dx}, \quad A_1 = x, \quad E = -a. \quad (\text{B.1})$$

This solution represents an infinitely long static D-string.

(ii) $\lambda > 0$ and $\tilde{\lambda} = 0$

In the nontrivial irreducible representations, A_0 , A_1 and E are operators acting on $\mathcal{L}^2(\mathbb{R})$, which are given by

$$A_0 = -i\sqrt{\lambda}\frac{d}{dx}, \quad A_1 = a \cosh x + b \sinh x, \quad E = -\sqrt{\lambda}(a \sinh x + b \cosh x), \quad (\text{B.2})$$

where $a, b \in \mathbb{R}$.

For $a = b$, the solution reduces to $A_1 = a \exp(x)$ and $E = \sqrt{\lambda}A_1$, which corresponds to the $d = 1$ case of (3.37). We can construct an $SO(3)$ symmetric solution from this case by the aforementioned procedure. The solution thus obtained is equivalent to the one we constructed from the Lie algebra (3.37) with $d = 3$ in ref. [38], which represents a $(3 + 1)$ -dimensional expanding universe. For $a \neq b$, we can also construct an $SO(3)$ symmetric solution in the same way. We have checked numerically that the resulting solution exhibits essentially the same behavior as the one for $a = b$.

(iii) $\lambda < 0$ and $\tilde{\lambda} = 0$

In the nontrivial irreducible representations, A_0 , A_1 and E are operators acting on $\mathcal{L}^2(0, 2\pi)$, which are given as

$$A_0 = -i\sqrt{-\lambda}\frac{d}{dx}, \quad A_1 = a \cos x + b \sin x, \quad E = \sqrt{-\lambda}(a \sin x - b \cos x). \quad (\text{B.3})$$

⁶The five cases (i)~(v) correspond to $A_{3,1}$, $A_{3,4}$, $A_{3,6}$, $A_{3,6}$ and $A_{3,4}$, respectively, in Table I of ref. [44]. In particular, $A_{3,1}$ is a nilpotent Lie algebra.

This case is analytically tractable. For the SO(3) symmetric solution obtained from the Lie algebra (B.3), we can define $R(t)$ as in the case of the SU(2) and SU(1, 1) solutions. We find that $R(t) = \text{constant}$.

(iv) $\lambda = 0$ and $\tilde{\lambda} > 0$

In the nontrivial irreducible representations, A_0 , A_1 and E are operators acting on $\mathcal{L}^2(0, 2\pi)$, which are given as

$$A_0 = a \cos x + b \sin x, \quad A_1 = -i\sqrt{\tilde{\lambda}} \frac{d}{dx}, \quad E = \sqrt{\tilde{\lambda}}(-a \sin x + b \cos x). \quad (\text{B.4})$$

We have seen numerically that the SO(3) symmetric solution obtained from (B.4) exhibits almost the same behavior as the one in (iii).

(v) $\lambda = 0$ and $\tilde{\lambda} < 0$

In the nontrivial irreducible representations, A_0 , A_1 and E are operators acting on $L^2(\mathbb{R})$:

$$A_0 = a \cosh x + b \sinh x, \quad A_1 = -i\sqrt{-\tilde{\lambda}} \frac{d}{dx}, \quad E = \sqrt{-\tilde{\lambda}}(a \sinh x + b \cosh x). \quad (\text{B.5})$$

We have seen numerically that $R(t)$ in the SO(3) symmetric solution obtained from (B.5) exhibits a behavior different from (ii). The solutions for $b = 0$ has an expanding regime only, whereas the solutions for $a = 0$ have both expanding and contracting regimes.

C. Examples of classical solutions describing noncommutative space

In this appendix we present some examples of classical solutions which are not manifestly space-space commutative. These solutions are based on the Lie algebras SO(6), SO(5, 1) and SO(4, 2), and we interpret them as describing (3+1)-dimensional universes with SO(4) symmetry. These Lie algebras obey the commutation relations

$$[L_{\alpha\beta}, L_{\gamma\delta}] = ig_{\alpha\gamma}L_{\beta\delta} + ig_{\beta\delta}L_{\alpha\gamma} - ig_{\alpha\delta}L_{\beta\gamma} - ig_{\beta\gamma}L_{\alpha\delta}, \quad (\text{C.1})$$

where $\alpha, \beta, \gamma, \delta = 1, 2, \dots, 6$. The non-vanishing components of $g_{\alpha\beta}$ are

$$\begin{aligned} g_{ii} &= 1 \quad (i = 1, 2, 3, 4), \\ g_{55} &= \begin{cases} 1 & \text{for SO(6), SO(5, 1)} , \\ -1 & \text{for SO(4, 2)} , \end{cases} \\ g_{66} &= \begin{cases} 1 & \text{for SO(6)} , \\ -1 & \text{for SO(5, 1), SO(4, 2)} . \end{cases} \end{aligned} \quad (\text{C.2})$$

We set

$$\begin{aligned} A_0 &= aL_{56}, \\ A_i &= bL_{5i} \quad (i = 1, 2, 3, 4), \\ A_5, \dots, A_9 &= 0. \end{aligned} \quad (\text{C.3})$$

Then it is easy to verify that the equations (3.2) and (3.3) are satisfied if

$$\begin{aligned}\lambda &= -a^2 g_{55} g_{66} + 4b^2 g_{55} \ , \\ \tilde{\lambda} &= 4b^2 g_{55} \ .\end{aligned}\tag{C.4}$$

References

- [1] H. Liu, G. W. Moore and N. Seiberg, *Strings in a time dependent orbifold*, JHEP **06** (2002) 045 [[hep-th/0204168](#)].
- [2] H. Liu, G. W. Moore and N. Seiberg, *Strings in time dependent orbifolds*, JHEP **10** (2002) 031 [[hep-th/0206182](#)].
- [3] A. Lawrence, *On the instability of 3-D null singularities*, JHEP **11** (2002) 019 [[hep-th/0205288](#)].
- [4] G. T. Horowitz and J. Polchinski, *Instability of space - like and null orbifold singularities*, Phys. Rev. D **66** (2002) 103512 [[hep-th/0206228](#)].
- [5] M. Berkooz, B. Craps, D. Kutasov and G. Rajesh, *Comments on cosmological singularities in string theory*, JHEP **03** (2003) 031 [[hep-th/0212215](#)].
- [6] T. Banks, W. Fischler, S. H. Shenker, and L. Susskind, *M theory as a matrix model: a conjecture*, Phys. Rev. D **55** (1997) 5112 [[arXiv:hep-th/9610043](#)].
- [7] N. Ishibashi, H. Kawai, Y. Kitazawa, and A. Tsuchiya, *A large- N reduced model as superstring*, Nucl. Phys. B **498** (1997) 467 [[arXiv:hep-th/9612115](#)].
- [8] R. Dijkgraaf, E. P. Verlinde and H. L. Verlinde, *Matrix string theory*, Nucl. Phys. B **500** (1997) 43 [[hep-th/9703030](#)].
- [9] D. Z. Freedman, G. W. Gibbons, and M. Schnabl, *Matrix cosmology*, AIP Conf. Proc. **743** (2005) 286 [[arXiv:hep-th/0411119](#)].
- [10] B. Craps, S. Sethi, and E. P. Verlinde, *A matrix big bang*, JHEP **10** (2005) 005 [[arXiv:hep-th/0506180](#)].
- [11] M. Li, *A class of cosmological matrix models*, Phys. Lett. B **626** (2005) 202 [[arXiv:hep-th/0506260](#)].
- [12] S. R. Das and J. Michelson, *PP wave big bangs: matrix strings and shrinking fuzzy spheres*, Phys. Rev. D **72** (2005) 086005 [[arXiv:hep-th/0508068](#)].
- [13] B. Chen, *The time-dependent supersymmetric configurations in M-theory and matrix models*, Phys. Lett. B **632** (2006) 393 [[arXiv:hep-th/0508191](#)].
- [14] J. H. She, *A matrix model for Misner universe*, JHEP **01** (2006) 002 [[arXiv:hep-th/0509067](#)].
- [15] E. J. Martinec, D. Robbins and S. Sethi, *Toward the end of time*, JHEP **08** (2006) 025 [[arXiv:hep-th/0603104](#)].
- [16] T. Ishino and N. Ohta, *Matrix string description of cosmic singularities in a class of time-dependent solutions*, Phys. Lett. B **638** (2006) 105 [[arXiv:hep-th/0603215](#)].
- [17] T. Matsuo, D. Tomino, W. Y. Wen and S. Zeze, *Quantum gravity equation in large N Yang-Mills quantum mechanics*, JHEP **11** (2008) 088 [[arXiv:0807.1186](#)].

- [18] D. Klammer and H. Steinacker, *Cosmological solutions of emergent noncommutative gravity*, Phys. Rev. Lett. **102** (2009) 221301 [[arXiv:0903.0986](#)].
- [19] J. Lee and H. S. Yang, *Quantum gravity from noncommutative spacetime*, [arXiv:1004.0745](#).
- [20] H. Aoki, S. Iso, H. Kawai, Y. Kitazawa, and T. Tada, *Space-time structures from IIB matrix model*, Prog. Theor. Phys. **99** (1999) 713 [[hep-th/9802085](#)].
- [21] T. Hotta, J. Nishimura and A. Tsuchiya, *Dynamical aspects of large N reduced models*, Nucl. Phys. B **545** (1999) 543 [[hep-th/9811220](#)].
- [22] J. Ambjorn, K. N. Anagnostopoulos, W. Bietenholz, T. Hotta and J. Nishimura, *Large N dynamics of dimensionally reduced 4-D $SU(N)$ super Yang-Mills theory*, JHEP **07** (2000) 013 [[hep-th/0003208](#)].
- [23] J. Ambjorn, K. N. Anagnostopoulos, W. Bietenholz, T. Hotta and J. Nishimura, *Monte Carlo studies of the IIB matrix model at large N* , JHEP **07** (2000) 011 [[hep-th/0005147](#)].
- [24] K. N. Anagnostopoulos and J. Nishimura, *New approach to the complex-action problem and its application to a nonperturbative study of superstring theory*, Phys. Rev. D **66** (2002) 106008.
- [25] K. N. Anagnostopoulos, T. Azuma and J. Nishimura, *A practical solution to the sign problem in a matrix model for dynamical compactification*, JHEP **10** (2011) 126 [[arXiv:1108.1534](#)].
- [26] J. Nishimura and G. Vernizzi, *Spontaneous breakdown of Lorentz invariance in IIB matrix model*, JHEP **04** (2000) 015. [[arXiv:hep-th/0003223](#)].
- [27] J. Nishimura and G. Vernizzi, *Brane world from IIB matrices*, Phys. Rev. Lett. **85** (2000) 4664 [[hep-th/0007022](#)].
- [28] J. Nishimura and F. Sugino, *Dynamical generation of four-dimensional space-time in the IIB matrix model*, JHEP **0205** (2002) 001 [[hep-th/0111102](#)].
- [29] H. Kawai, S. Kawamoto, T. Kuroki, T. Matsuo and S. Shinohara, *Mean field approximation of IIB matrix model and emergence of four-dimensional space-time*, Nucl. Phys. B **647** (2002) 153 [[hep-th/0204240](#)].
- [30] T. Aoyama and H. Kawai, *Higher order terms of improved mean field approximation for IIB matrix model and emergence of four-dimensional space-time*, Prog. Theor. Phys. **116** (2006) 405 [[hep-th/0603146](#)].
- [31] T. Imai, Y. Kitazawa, Y. Takayama and D. Tomino, *Quantum corrections on fuzzy sphere*, Nucl. Phys. B **665** (2003) 520 [[hep-th/0303120](#)].
- [32] T. Imai, Y. Kitazawa, Y. Takayama and D. Tomino, *Effective actions of matrix models on homogeneous spaces*, Nucl. Phys. B **679** (2004) 143 [[hep-th/0307007](#)].
- [33] T. Imai and Y. Takayama, *Stability of fuzzy $S^2 \times S^2$ geometry in IIB matrix model*, Nucl. Phys. B **686** (2004) 248 [[hep-th/0312241](#)].
- [34] J. Nishimura, T. Okubo and F. Sugino, *Systematic study of the $SO(10)$ symmetry breaking vacua in the matrix model for type IIB superstrings*, JHEP **10** (2011) 135 [[arXiv:1108.1293](#)].
- [35] W. Krauth, H. Nicolai, and M. Staudacher, *Monte Carlo approach to M theory*, Phys. Lett. B **431** (1998) 31.

- [36] P. Austing and J. F. Wheeler, *Convergent Yang-Mills matrix theories*, JHEP **04** (2001) 019.
- [37] S. -W. Kim, J. Nishimura and A. Tsuchiya, *Expanding (3+1)-dimensional universe from a Lorentzian matrix model for superstring theory in (9+1)-dimensions*, Phys. Rev. Lett. **108** (2012) 011601 [[arXiv:1108.1540](#)].
- [38] S. -W. Kim, J. Nishimura and A. Tsuchiya, *Expanding universe as a classical solution in the Lorentzian matrix model for nonperturbative superstring theory*, Phys. Rev. D **86** (2012) 027901 [[arXiv:1110.4803](#)].
- [39] M. Fukuma, H. Kawai, Y. Kitazawa and A. Tsuchiya, *String field theory from IIB matrix model*, Nucl. Phys. B **510** (1998) 158 [[hep-th/9705128](#)].
- [40] J. Ambjorn, J. Jurkiewicz and R. Loll, *Reconstructing the universe*, Phys. Rev. D **72** (2005) 064014 [[hep-th/0505154](#)].
- [41] H. Kawai and T. Okada, *Asymptotically vanishing cosmological constant in the multiverse*, Int. J. Mod. Phys. A **26** (2011) 3107 [[arXiv:1104.1764](#)].
- [42] H. Steinacker, *Split noncommutativity and compactified brane solutions in matrix models*, Prog. Theor. Phys. **126** (2012) 613 [[arXiv:1106.6153](#)].
- [43] A. Chatzistavrakidis, *On Lie-algebraic solutions of the type IIB matrix model*, Phys. Rev. D **84** (2011) 106010 [[arXiv:1108.1107](#)].
- [44] J. Patera, R. T. Sharp, P. Winternitz, H. Zassenhaus, *Invariants of real low dimension Lie algebras*, J. Math. Phys. **17** (1976) 986.
- [45] J. Nishimura, *The origin of space-time as seen from matrix model simulations*, [arXiv:1205.6870](#), to appear in Prog. Theor. Exp. Phys.
- [46] M. Hanada, H. Kawai and Y. Kimura, *Describing curved spaces by matrices*, Prog. Theor. Phys. **114** (2006) 1295 [[hep-th/0508211](#)].
- [47] A. Chatzistavrakidis, H. Steinacker and G. Zoupanos, *Intersecting branes and a standard model realization in matrix models*, JHEP **09** (2011) 115 [[arXiv:1107.0265](#)].
- [48] H. Aoki, *Chiral fermions and the standard model from the matrix model compactified on a torus*, Prog. Theor. Phys. **125** (2011) 521 [[arXiv:1011.1015](#)].
- [49] N. Ja. Vilenkin and A. U. Klimyk, *Representation of Lie groups and special functions volume 1*, Kluwer Academic Publishers (1991).